**Lorentz Coordinate Transformations (frames S and S')**

$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$
$$ct' = \gamma(ct - \beta x)$$
$$\beta = v/c \quad , \quad \gamma = (1-\beta^2)^{-1/2}$$

or

$$dx' = \gamma(dx - \beta cdt)$$
$$dy' = dy$$
$$dz' = dz$$
$$cdt' = \gamma(cdt - \beta dx)$$

**Lorentz Velocity Transformations (frames S and S')**

$$u'_x = c\frac{dx'}{cdt'} = \frac{\gamma(dx - \beta cdt)}{\gamma(cdt - \beta dx)}c = \frac{\frac{dx}{dt} - \beta c}{1 - \frac{\beta\frac{dx}{dt}}{c}} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u'_y = c\frac{dy'}{cdt'} = \frac{u_y}{\beta\left(1 - \frac{u_x v}{c^2}\right)}$$

$$u'_z = c\frac{dz'}{cdt'} = \frac{u_z}{\beta\left(1 - \frac{u_x v}{c^2}\right)}$$

**Lorentz Acceleration Transformations (frames S and S')**

Using inverse transformations

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \quad , \quad cdt = \gamma(cdt' + \beta dx')$$

we have

$$\frac{du_x}{dt} = \frac{1}{\beta^3\left(1 + \frac{u'_x v}{c^2}\right)^3}\frac{du'_x}{dt}$$

$$\frac{du_y}{dt} = \frac{1}{\beta^3\left(1 + \frac{u'_x v}{c^2}\right)^3}\frac{du'_y}{dt} - \frac{vu'_y}{c^2\beta^3\left(1 + \frac{u'_x v}{c^2}\right)^3}\frac{du'_x}{dt}$$

$$\frac{du_z}{dt} = \frac{1}{\beta^3\left(1 + \frac{u'_x v}{c^2}\right)^3}\frac{du'_z}{dt} - \frac{vu'_z}{c^2\beta^3\left(1 + \frac{u'_x v}{c^2}\right)^3}\frac{du'_x}{dt}$$

Clearly, acceleration is NOT an invariant in special relativity.

However, these results show that acceleration is an absolute quantity, that is, ALL observers agree whether a body is accelerating or not. In other words, if the acceleration is zero in one frame, then it is necessarily zero in any other frame.

Summarizing, we have this table

| Theory | Position | Velocity | Time | Acceleration |
|---|---|---|---|---|
| Newtonian | Relative | Relative | Absolute | Absolute |
| Special Relativity | Relative | Relative | Relative | Absolute |
| General Relativity | Relative | Relative | Relative | Relative |

**Uniform Acceleration**

The Newtonian definition of a particle moving under **uniform** acceleration is

$$\frac{du}{dt} = \text{constant}$$

This turns out to be inappropriate in special relativity since it would imply that $u \to \infty$ as $t \to \infty$, which we know is impossible.

We therefore adopt a different definition.

Acceleration is said to be **uniform** in special relativity if it has the same value in any **co-moving frame**, that is, at each instant, the acceleration in an inertial frame traveling with the same velocity as the particle has the same value.

This is analogous to the idea in Newtonian theory of "motion under a constant force". For example, a spaceship whose motor is set at a constant emission rate would be uniformly accelerated in this sense.

Taking the velocity of the particle to be $u = u(t)$ relative to an inertial frame S, then at any instant in a co-moving frame S', it follows that $v = u$, the velocity relative to S' is zero, i.e., $u' = 0$, and the acceleration is a constant, $a$ say, i.e.,

$$\frac{du'}{dt'} = a$$

We then have (considering only 1-dimensional motion)

$$\frac{du}{dt} = \frac{1}{\beta^3\left(1 + \frac{u'v}{c^2}\right)^3}\frac{du'}{dt} = \frac{a}{\beta^3} = \left(1 - \frac{u^2}{c^2}\right)^{3/2} a$$

We can solve this differential equation by separating the variables

$$\frac{du}{\left(1-\frac{u^2}{c^2}\right)^{3/2}} = a\,dt$$

and integrating both sides. Assuming that the particle starts from rest at $t = t_0$, we find

$$\frac{u}{\left(1-\frac{u^2}{c^2}\right)^{1/2}} = a(t-t_0)$$

Solving for $u$ we get

$$u = \frac{dx}{dt} = \frac{a(t-t_0)}{\left[1+\frac{a^2(t-t_0)^2}{c^2}\right]^{1/2}}$$

Now, integrating with respect to $t$, and setting $x = x_0$ at $t = t_0$, we get

$$(x-x_0) = \frac{c}{a}\left[c^2 + a^2(t-t_0)^2\right]^{1/2} - \frac{c^2}{a}$$

This can be rewritten in the form

$$\frac{\left(x - x_0 + \frac{c^2}{a}\right)^2}{\left(\frac{c^2}{a}\right)^2} - \frac{(ct-ct_0)^2}{\left(\frac{c^2}{a}\right)^2} = 1$$

which is a **hyperbola** in $(x,ct)$ space.

If, in particular, we take $x_0 - \frac{c^2}{a} = t_0 = 0$ we have

$$x^2 - c^2t^2 = \frac{c^2}{a}$$

which is a family of hyperbolas for different values of the acceleration $a$. These world-lines are known as **hyperbolic motions.**

The **radar distance** between the world-lines is **constant.**