Physics 6H Freshman Seminar Fall 2004 Assignment #9

Readings:

- Boccio - Schrodinger Equation
- Boccio - Simple Quantum Systems
- Boccio - Uncertainty Principle

Summary: This week we derive the Scrodinger equation, develop procedures to deal with time evolution of quantum properties, develop solutions for simple quantum systems and present the Heisenberg Uncertainty Principle (HUP).

Many of the problems this week are standard quantum mechanics problems. They are not easy.

Everyone Problems:

EP-38 Angular Momentum
EP-40 Atoms and Angular Momentum
EP-56 Time development
EP-68 Atom near a wall

Individual Problems:

EP-39 Angular Momentum Eigenvectors
EP-57 Time development redux
EP-61 Time Evolution redux
EP-62 Time Evolution again
EP-69 Particle in a box with a membrane
EP-71 Harmonic oscillator

Presentations:

- Time Evolution in the Schrodinger Picture
- Schrodinger Wave Equation
- More Time Development
- Boundary Conditions
- Infinite Square Well
- Tunneleing Through a Barrier
- Finite Square Well
- Delta Function Potentials
- Heisenberg Uncertainty Principle

Seminar Break:

Extra Problems:

EP-38. Angular Momentum - When we measure the angular momentum of some atoms along a given direction in space (see Stern-Gerlach discussions), say the z-axis, we get three values $(\hbar, 0, -\hbar)$.

(a) Construct an operator for angular momentum in the z-direction (call it $\hat{O}_{L_z}$) using the most simple basis states that are eigenstates of this operator.

When we measure the magnitude squared of the angular momentum of these same atoms, we always get $\hbar^2$. Also, measurements of the z-component and magnitude squared can be done in any order and the results are the same.

(b) Construct an operator for the magnitude squared of the angular momentum (call it $\hat{O}_{L^2}$) using the same basis states you used for $\hat{O}_{L_z}$.

(c) In this basis, the operator for the x-component of angular momentum is

$$\hat{O}_{L_x} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Are the eigenstates of $\hat{O}_{L_z}$ also eigenstates of $\hat{O}_{L_x}$?

(d) Do $\hat{O}_{L_z}$ and $\hat{O}_{L_x}$ commute?

(e) What can you say about successive measurements of the z- and x-components of the angular momentum?

EP-39. Angular Momentum Eigenvectors - Using the eigenstates of the $\hat{O}_{L_z}$ operator found in Problem 38 as the basis states, find the eigenvalues and eigenvectors of the $\hat{O}_{L_x}$ operator.

EP-40.Atoms and Angular Momentum - Consider an atom in the following state

$$|\psi\rangle = \frac{1}{\sqrt{14}}\left(|1\rangle + 2|2\rangle + 3i|3\rangle\right)$$

where $|1\rangle$ is the eigenstate of $\hat{O}_{L_z}$ with eigenvalue $\hbar$, $|2\rangle$ is the eigenstate of $\hat{O}_{L_z}$ with eigenvalue 0 and $|3\rangle$ is the eigenstate of $\hat{O}_{L_z}$ with eigenvalue $-\hbar$.

(a) If the z-component of angular momentum is measured for an atom in the state $|\psi\rangle$, what are the possible results and with what probabilities do they occur?

(b) If the x-component of angular momentum is measured for an atom in the state $|\psi\rangle$, what are the possible results and with what probabilities do they occur?

EP-56. Time development Let the energy operator for a three state system be

$$\hat{H} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}$$

where the basis being used are the energy eigenstates labelled by $|1\rangle$, $|2\rangle$, and $|3\rangle$.

(a) If the state of the system at $t=0$ is $|\psi(0)\rangle = |2\rangle$, what is the state of the system at time t later $|\psi(t)\rangle$?

(b) If the state of the system at $t=0$ is $|\psi(0)\rangle = |3\rangle$, what is the state of the system at time t later $|\psi(t)\rangle$?

(c) If the state of the system at $t=0$ is $|\psi(0)\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle + \frac{1}{2}|3\rangle$, what is the state of the system at time t later $|\psi(t)\rangle$?

EP-57 - Time development redux

Consider an energy operator $\hat{H}$ with three eigenvectors given by the equations

$$\hat{H}|E=+10\rangle = +|E=+10\rangle$$

$$\hat{H}|E=-10\rangle = -|E=-10\rangle$$

where $\{|E=+10\rangle, |E=-10\rangle\}$ form an orthonormal basis.

Now suppose that we are investigating some physical system and want to predict future value of a "spin" operator $\hat{S}$ with eigenvectors

$$\hat{S}|S=+1\rangle = \hat{S}\left(\frac{1}{\sqrt{2}}\left[|E=+10\rangle + |E=-10\rangle\right]\right) = +\left(\frac{1}{\sqrt{2}}\left[|E=+10\rangle + |E=-10\rangle\right]\right)$$

$$\hat{S}|S=-1\rangle = \hat{S}\left(\frac{1}{\sqrt{2}}\left[|E=+10\rangle - |E=-10\rangle\right]\right) = -\left(\frac{1}{\sqrt{2}}\left[|E=+10\rangle - |E=-10\rangle\right]\right)$$

where the eigenvectors of the "spin" operator have been written in the "energy" basis.

Suppose, in addition, that a physical system is initially prepared to be in the state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|E=+10\rangle + |E=-10\rangle\right] = |S=1\rangle$$

that is, in an eigenvector of the "spin" operator.

(a) Is this state an eigenvector of the energy operator?

What is the probability that we will measure energy values $+10$ *or* -10 in this initial state? What is the probability that we will measure spin values $+1$ *or* -1 in this initial state ?

The time development operator is given by

$$\hat{T} = e^{-i\hat{H}t/\hbar}$$

(b) Determine

$$\hat{T}|E=+10\rangle = ? \qquad \hat{T}|S=+1\rangle = ?$$

$$\hat{T}|E=-10\rangle = ? \qquad \hat{T}|S=-1\rangle = ?$$

(c) Using (b) determine $|\psi(t)\rangle = \hat{T}|\psi(0)\rangle$ = the state of the system at a later time.

What is the probability that we will measure spin values $+1$ *or* -1 at this later time? You must calculate the quantities

$$P(S=1;t) = \left|\langle S=1|\psi(t)\rangle\right|^2 = \left|\langle S=1|\hat{T}|\psi(0)\rangle\right|^2$$

$$P(S=-1;t) = \left|\langle S=-1|\psi(t)\rangle\right|^2 = \left|\langle S=-1|\hat{T}|\psi(0)\rangle\right|^2$$

Will the spin value ever "flip" from $S=+1$ to $S=-1$?

EP-61. Time Evolution redux - Imagine a situation in which there are three energy states $|1\rangle$ with energy $E_1 = 1\,eV$, $|2\rangle$ with energy $E_2 = 2\,eV$ and $|3\rangle$ with energy $E_3 = 4\,eV$. Let the state of the system at $t=0$ be $|\psi(0)\rangle = \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

(a) What are the possible results of an energy measurement on this system at $t=0$ and with what probabilities will each of them occur?

(b) What are the possible results of an energy measurement on this system at $t = 5\times10^{-14}\,\mathrm{sec}$ and with what probabilities will each of them occur?

(c) Are you surprised by your answer? Can you make sense of it?

EP-62. Time Evolution again - A state $|X\rangle$ is given in the hardness basis as

$$|X\rangle = \frac{1}{3}|h\rangle + \frac{i2\sqrt{2}}{3}|s\rangle$$

(a) What are the possible outcomes and probabilities for measurement of hardness and color on this state at $t=0$?

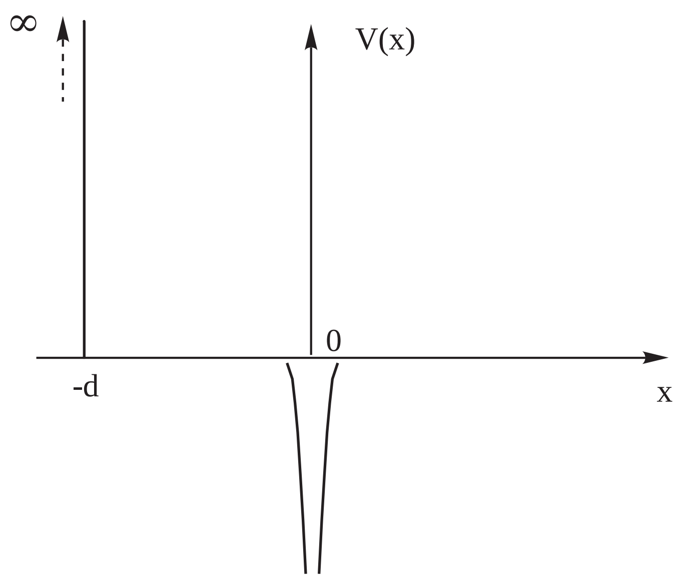
(b) Imagine that due to an external field, the two color states $|g\rangle$ and $|m\rangle$, are eigenstates of the energy operator with the energy of the green state being zero and the energy of the magenta state being $1.0\times10^{-34}\,J$. What are the possible outcomes and probabilities for measurement of hardness and color on this state after 1 second?

EP-68. Atom near a wall

An approximate model for an atom near a wall is to consider a particle moving under the influence of the one-dimensional potential given by

$$V(x) = \begin{cases} -V_0\delta(x) & x > -d \\ \infty & x < -d \end{cases}$$

as shown below:



(a) Find the modification of the bound-state energy caused by the wall when it is "far away". Define what you mean by "far away".
(b) What is the exact condition on V_0 and d for the existence of at least one bound state ?

EP-69. Particle in a box with a membrane
Do matrix solution and pure Dirac algebra solution.

A box, containing a particle, is divided into a right and a left compartment by a thin partition. Suppose that the amplitude for the particle being on the left side of the box is ψ_1 and the amplitude for the particle being on the right side of the box is ψ_2 . Neglect spatial variations of these amplitudes within the halves of the box. Suppose that the particle can tunnel through the partition and that the rate of change of the amplitude on the right is given by

$$i\hbar\frac{\partial\psi_2}{\partial t} = K\psi_1$$

where K is real. Assume that in the absence of tunneling, i.e., an impermeable membrane, that $\frac{\partial\psi_1}{\partial t} = 0$.

(a) What is the equation that determines the rate of change of the amplitude on the left?
(b) Find the normalized energy eigenstates (2-component vectors) of the particle in the box. Have these states definite parities?
(c) Suppose that at time t = 0, the amplitude on the right equals $e^{i\delta}$ time the amplitude on the left. Calculate, as a function of time, the time rate of change of the probability of observing the particle on the left.

EP-71. Harmonic oscillator - The text and extra notes discussed the infinite dimension linear state space associated with position along one direction in space (the x-axis for example), The wave function

$\psi(x)$ was defined as the function of x that gives the coefficient (component) for the basis state for each value of x. In the notes we solve some 1-dimensional problems (infinite square well, finite square well, delta unction and barrier). Let us now look at the quantum mechanical harmonic oscillator. The wave function must satisfy the time-independent Schrodinger wave equation with a potential energy function $V(x) = \frac{1}{2}kx^2$, that is,

$$\langle x|\hat{H}|\psi\rangle = \langle x|E|\psi\rangle = E\langle x|\psi\rangle$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}kx^2\psi(x) = E\psi(x)$$

The ground-state(lowest energy level) wavefunction is

$$\psi(x) = \frac{1}{\sqrt{a\sqrt{\pi}}}e^{-\frac{x^2}{2a^2}} \quad , \quad a^2 = \frac{\hbar}{\sqrt{mk}}$$

how that $\psi(x)$ is a solution to the Schrodinger equation and determine the ground-state energy.