Numerical study of rice-pile model.

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A one-dimensional model of a rice-pile is numerically studied for different driving mechanisms. We found that for a sufficiently large system, there is a sharp transition between the trivial behaviour of a 1D BTW model and self-organized critical (SOC) behaviour. Depending on the driving mechanism, the self-organized critical rice-pile model belongs to two different universality classes.

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I. INTRODUCTION

In their pioneer work Bak, Tang and Wiensenfeld [1] introduced the concept of self-organized criticality (SOC) to describe the behaviour of extended dissipative systems. The paradigm of SOC is an idealized sandpile where grains added to a pile dissipate their potential energy through avalanches with no characteristic scale [1-10]. Besides the numerical simulations many different methods were also used to treat the SOC problems. Dynamical mean-field theory [11] gives a unified description of some stochastic SOC systems including the BTW sandpile model and the forest fire model [12]. Langevin type approaches [13] have been used on a phenomenological basis. Furthermore, a real space renormalization group method [14] provided good estimates of the exponents. Early experimental studies of real sandpiles lead to clear disagreement with the numerical models: Bounded distributions of avalanche sizes were observed instead of the expected power-law behavior [15-19]. Using grains of rice, Frette et al. [20] showed that the dynamics exhibit self-organized critical behaviour in one case (for grains with a large aspect ratio) but not in another (for less elongated grains). To take into account the changes in the local slopes observed in the rice-pile experiment, Christensen et al. proposed a rice-pile model, hereafter called Oslo model [21-23], where the critical slope for each site is a dynamical variable. Here we propose a model for a pile of granular material where we introduce randomness in the relaxation rule and use two types of driving mechanisms: fixed-position driving where the grains are added on the top of the pile, and random-position driving where the grains are added on randomly chosen sites. On the one hand, this model permits to investigate the transition between the 1D BTW model and the rice-pile model, and on the other hand to study the effect of the driving mechanisms on the size and transit-time avalanche exponents.
This paper is a review of studies published in references 24 and 25.

II. MODEL

Our model is defined as follows: Consider the one-dimensional system formed by a set of L sites, labeled by integers $i=1$ to L, with a wall at $i=0$ and an open boundary at $i=L+1$. An integer variable $h(i)$ is assigned to each site i and called the local height of the rice-pile at site i. Its local slope is defined as $z(i)=h(i)-h(i+1)$ for $i=1,2,...,L$. Initially, the system is empty, i.e., $h(i) = 0\ \forall i$.

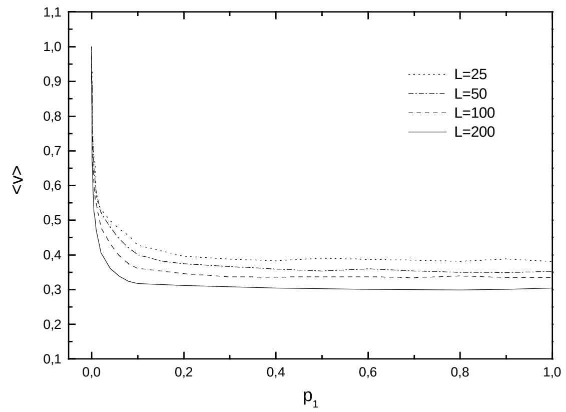

Fig. 1: The average transportation <v> as a function of the probability p_1 for several values of the system size and in the case of random driving mechanism.

The profile of the pile evolves through two mechanisms, perturbation and relaxation, with a separation in time scales i.e. the rate of deposition is slow enough that any avalanche, triggered by a deposited grain, will have ended before a new grain is added. At each time step, a grain is added to a column i:

$$h(i) \rightarrow h(i) + 1, \qquad (1)$$

if at a certain site i, the local slope $z(i)$ is greater than a threshold $z^c(i)$ $(z(i) > z^c(i))$, where $z^c(i)$ takes randomly a value 1 or 2, then this site topples with a probability which depends on its slope $z(i)$, namely: If $z(i) = 2$, the site i topples with a probability p_1 and if $z(i)>2$, it topples with probability p_2, then one grain of rice

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will be transferred from a column i to its right neighbor according to the following equation

$$h(i) \to h(i) - 1$$
$$h(i+1) \to h(i+1) + 1, \qquad (2)$$

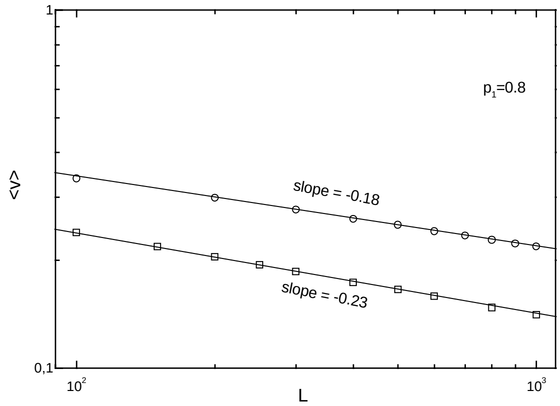


Fig. 2: The average velocity <*v*> as a function of the system size *L*, for $p_1 = 0.8$. The open squares correspond to the fixed-position driving. The open circles correspond to the random-position driving.

and all unstable sites topple in parallel. Notice that the model connects the probabilities of toppling, p_1 and p_2, to the actual slope $z(i) > z^c(i)$ assigned to each site i. Consequently, the model dœsn't allow for storage of externally imposed particles and so it can not display a subcritical behavior. If we set $p_1 = 0$ and $p_2 = 1$, the model becomes the BTW model and if we put $p_1 = p_2 = 1$, it is just the rice-pile model or Oslo rice-pile model if the driving mechanism is at the top of the pile. Furthermore, as in Ref. 21, at each grain is associated a transit time *T* defined by $T = T_{out} - T_{in}$, where T_{out} and T_{in} denote the output and the input time of the grain; the time is measured in the unit of grains addition. Thus, we measure the avalanche size (i.e. the number of toppling sites), the transit-time *T* and their corresponding distributions. In the next section, we will consider two ways of driving mechanisms: One way is to add grains to the top of the pile (site *i=1*) as it is usually made in the Oslo rice-pile model as well as in the experiment. Another way is to add grains to randomly chosen positions (random driving mechanism). This later way introduces an external stochasticity in our model. Before going ahead, let us remind some differences between the 1D BTW model and the Oslo rice-pile model. In the Oslo model, the randomness is internal and inherent in the dynamics. For arbitrary initial conditions, the system reaches a stationary state characterized by power laws. However, the 1D BTW, where the randomness is external and the critical slope $z^c(i)$ is constant, the system reaches a stationary trivial behavior.

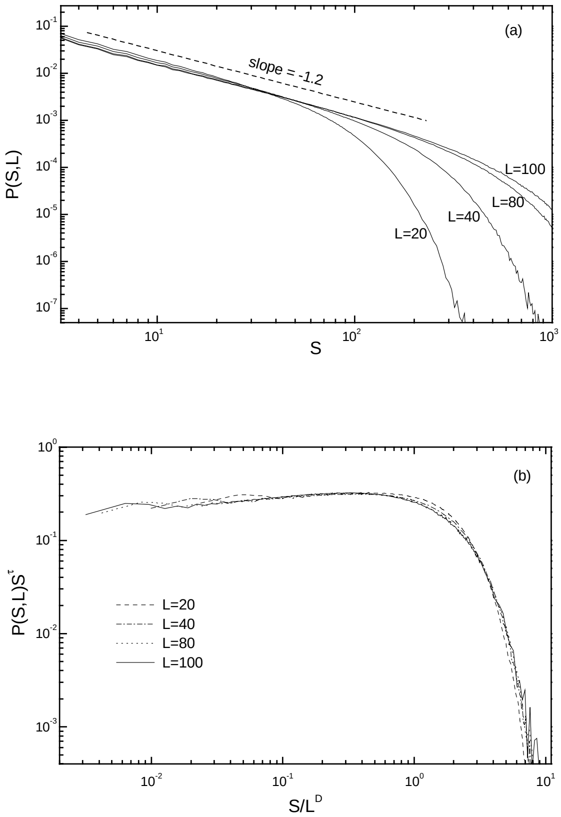


Fig. 3: (a) Log-Log plot of the avalanche size distribution for several values of the system size *L* with $p_1 = 0.8$ and in the case of random driving position. (b) Data collapse of the curves displayed in (a) according to eq. 3 with the exponents $\tau = 1.20, D = 1.25$.

III. NUMERICAL SIMULATIONS

We have performed extensive numerical simulations and investigated the effects of the parameter p_1 and of driving mechanisms on the behaviour of the system. In the following we will take $p_2 = 1$ and p_1 less than p_2 because higher is the slope, higher is the jump probability. So by varying p_1 from *0* to *1*, we can -in a continuous manner- change the model from the 1D BTW sandpile model to the rice-pile model. Let us first study the transport properties of the model when the external grains are added to randomly chosen positions. In Fig. 1, we show the average transportation velocity of grains, defined as <*v*>=*1*/<*T*>, as a function of p_1. For the case $p_1 = 0$ (the 1D BTW model), after a certain time a stationary state is reached where every newly-added grain will slip out the pile instantly, thus the transit time is *T=0*, and then the average velocity is infinite. By increasing p_1 larger than a some value, p_1^c, where p_1^c depends on the system size, the velocity <v> becomes constant and tend to 0 when L becomes sufficiently large. The numerical results make us to consider that $p_1^c \to 0$ as $L \to \infty$. For $p_1 = 0^+$, $< v >= 1$, independent of the system size. So there is a sharp transition from $< v >= \infty$ for $p_1 = 0$ to $< v >= 1$ for $p_1 = 0^+$. This transition can be understood by the following argument. It is clear that when p_1 is exactly 0, no newly-added grain will stay in the pile as

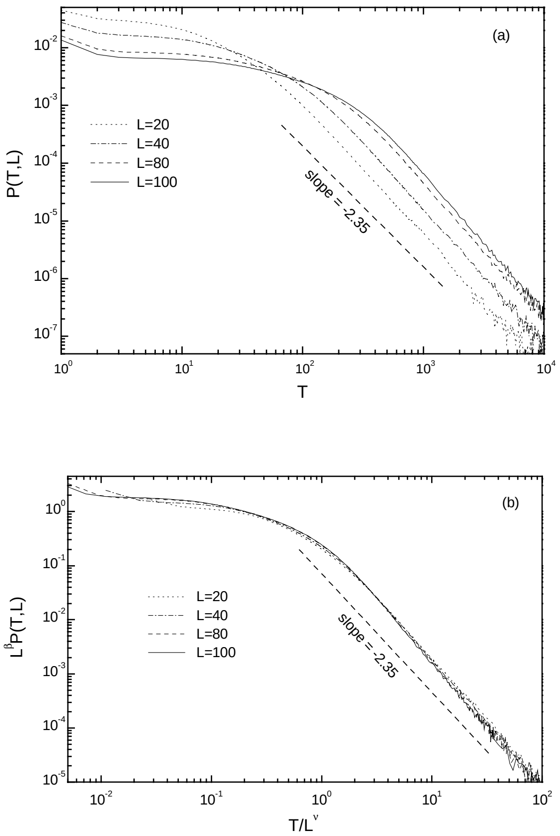


Fig. 4: Log-Log plot of the transit-time distribution for several values of the system size L with $p_1 = 0.8$ and in the case of random driving position. (b) Data collapse of the curves displayed in (a) according to eq. 4. The best fit to the numerical data gives the slope $\alpha = 2.35$.

long the stationary state is reached. So $T=0$ for every grain and hence $< T >= 0$. When $p_1 = 0^+$ some grains can be buried in the surface layer of the pile. These grains will stay in the pile for a very long time. Once they slip out of the pile, these grains, although very few in number, will make a significant contribution to $<T>$ since their transit times are extremely large. It is the existence of these grains that makes $< T >$ assume finite value for $p_1 = 0^+$. Between $p_1 = 0^+$ and $p_1 = p_1^c$, there is a crossover behaviour of $< v >$, which is due to finite size effects. Since we expect $p_1^c \to 0$ when $L \to \infty$, we can also expect that for infinite system the transition takes place at $p_1 = 0$ from $< v >= \infty$ to $< v >= 0$. Thus the sharp transition here is induced by tiny disorder. In Fig. 2, we plot the average velocity as a function of the system size for $p_1 = 0.8$ and for the two types of driving mechanisms. It is clear that for large system ($L>100$), the average velocity $< v >$ scales as $L^{-\gamma}$. When the grains are added on the top of the pile $\gamma = 0.23$ [24,25], while $\gamma = 0.18$ when the grains are added on a randomly chosen sites. Therefore, the average velocity decreases with the system size, which is due to the increase in the active zone depth with system size, as explained by Christensen et al. [21].

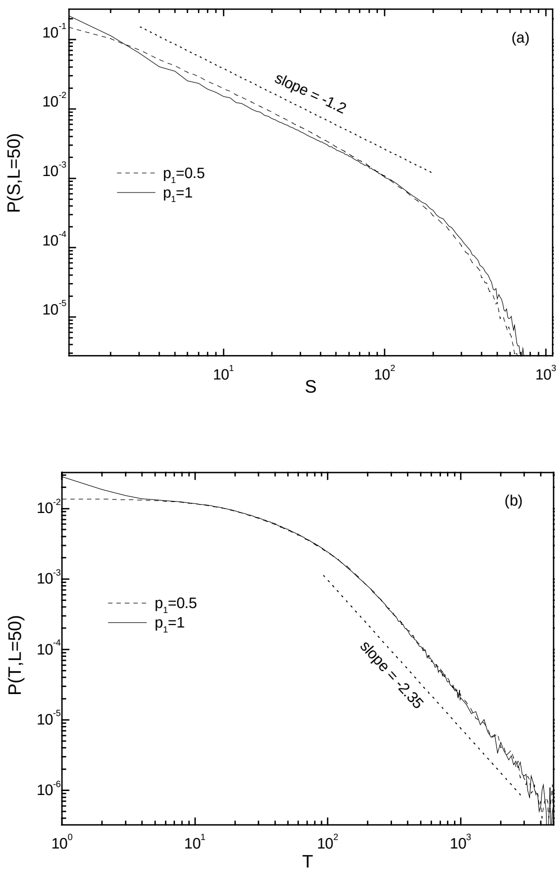
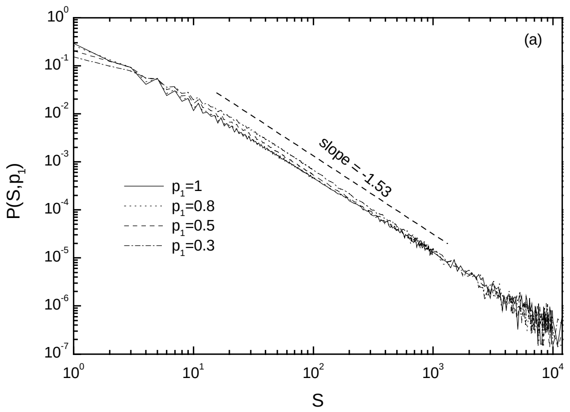


Fig. 5: (a) Log-Log plot of the avalanche size distribution in the case of random driving position. The best fit gives the slope $\tau = 1.20$. (b) Log-log plot of the transit time distribution in the case of random driving position. The best fit gives the slope $\alpha = 2.35$.

We have also studied the avalanche size and the transit-time distributions for different values of the probability $p_1 > p_1^c$ and for a random driving mechanism. In Fig. 3a we plot our simulation data for $p_1 = 0.8$ and for different system sizes. The distribution is a power law with the presence of a peak close to the cutoff size $S_c \propto L^D$. This is a finite-size effect which is due to the possibility to form a supercritical state which then relaxes through a very large avalanche. The distribution follows the scaling form:

$$P(S,L) = S^{-\tau} G(S / L^D), \qquad (3)$$

the best collapse is obtained with the exponents $\tau = 1.20 \pm 0.05$ and $D = 1.25 \pm 0.05$, cf. Fig. 3b. When the externally grains are added on the top of the pile, $\tau = 1.53 \pm 0.05$ and $D = 2.20 \pm 0.05$ [21,24].



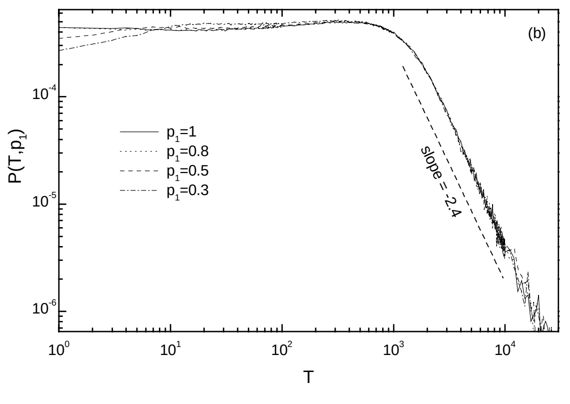


Fig. 6: (a) Log-log plot of the avalanche size distribution in the case of fixed-position driving. The best fit gives the slope $\tau = 1.53$. (b) Log-log plot of the transit time distribution in the case of fixed-position driving. The best fit gives the slope $\alpha = 2.40$. In both (a) and (b) the system size is *L=400*.

Thus by adding an external stochasticity (random driving mechanism) to the internal randomness (critical slope is a dynamical variable), the system belongs to another universality calss characterized by $\tau = 1.20 \pm 0.05$. By using that the average number of toppling is $< S >= L$ in the critical state, it follows from eq. 3 that: The relation

$$\tau = \frac{2D - 1}{D}, \qquad (4)$$

is in agreement with our numerical results. Furthermore, the distribution functions of transit times *P(T,L)* for several values of system sizes are shown in Fig. 4a. A data collapse for different system size *L* is obtained when plotting $L^\beta P(T,L)$ against the rescaled variable T / L^υ using $\upsilon = 1.20 \pm 0.15$ and $\beta = 1.20 \pm 0.15$; see Fig. 4b. Thus we can write:

$$P(T,L) = L^{-\beta} F(T / L^\upsilon), \qquad (5)$$

the scaling function *F* is of the form *F(x)=constant* for small *x* and $F(x) \propto x^{-\alpha}$ for larger *x*, υ is a critical exponent expressing how the crossover transit time T_c scales with the system size. The power-law exponent α for large transit time is obtained as $\alpha = 2.35$. Fig. 5a shows the avalanche-size distribution for the two values of probability p_1 greater than a critical value p_1^c. Fig. 5b gives the corresponding transit time distribution. It is clear that the size and the transit time exponents are insensitive to the values of the probability p_1 ($p_1 > p_1^c$).

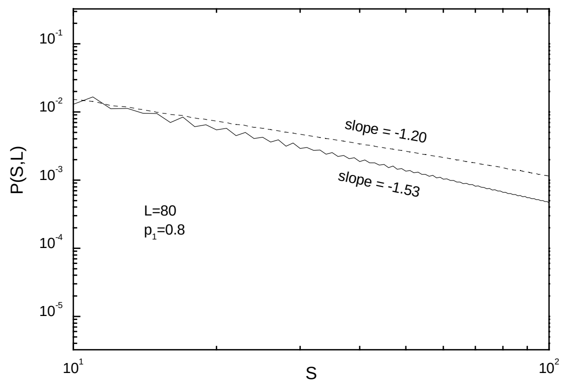


Fig. 7: Comparison of the avalanche size distributions for $p_1 = 0.8$. Solid line correspond to the fixed-position driving. Dashed line correspond to the random-position driving.

The same conclusion can be drawn in the case where the grains are added on the top of the pile, see Fig. 6a and Fig. 6b. Thus, the SOC state is insensitive to the variation in the jumping probability $p_1 > 0$. Finally, we could expect that a random external perturbation would lead to a decrease of the value of τ because by random driving we give less chance to big avalanches to occur. Fig. 7 gives the size distributions for the two types of driving mechanisms. From this figure one can clearly see that the exponents for differents driving mechanisms are different from each other.

IV CONCLUSION

In summary, we have investigated a one-dimensional rice pile model where the sites with higher slopes have more chance to topple (with a probability $p_2 = 1$) while the sites with lower slopes topple with a probability $p_1 \le p_2$. It is found that for a sufficiently large system, there is a sharp transition between the trivial behaviour and the SOC behaviour at $p_1 = 0$. In the case where the external grains are added to the top of the pile, the self-organized critical model belongs to the known universality class that is characterized by an avalanche exponent $\tau = 1.53 \pm 0.05$, whereas the model with random driving mechanism belongs to a new universality class characterized by $\tau = 1.20 \pm 0.05$.

ACKOWLEDGMENTS

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