

**READINGS:**

Boccio notes on chaos - pages 1-40  
Stewart - Chapters 1-8

**WEBSITE:** <http://chaos.swarthmore.edu/courses/SOC026/index.html>

**QUESTIONS FOR THOUGHT AND DISCUSSION AS YOU READ and IN CLASS:****DETERMINISTIC SYSTEMS, RANDOM PROCESSES, AND CHAOS ACCORDING TO IAN STEWART****A. General Considerations**

1. What should we understand as the meanings of the terms "deterministic systems," "random processes," and "chaos?"
2. What might be examples of a deterministic system, a random process, and chaos?
3. The book Chaos by James Gleick has the subtitle Making a New Science. Is the study of order and chaos a science? Or is the study of order and chaos a part of a particular scientific discipline such as astronomy, biology, mathematics, physics, or another discipline?

**B. Deterministic Systems in the Natural World**

1. What is the function of mathematics in the study of the natural world?
2. What does the discovery of "laws of nature" contribute to an understanding of the natural world?
3. In what ways are mathematical formulations of the laws of nature useful? In particular, for what purposes do we reduce the laws of nature to differential equations? (By the way, what is a differential equation? As a student of liberal learning, try to answer this question without making use of technical aspects of mathematics.)
4. What is the connection between the reduction of the laws of nature to differential equations and the conclusion that those laws describe deterministic systems?

**C. The Encounter of Voyager 1 and 2 with Hyperion.**

1. Give a precise description of the hypothetical experiment presented in the section "Voyage to Hyperion" in Does God Play Dice?
2. What is the expected result of that experiment? Why?
3. What is the "observed" result?
4. What is the apparent paradox revealed by a comparison of

the expected and observed results?

5. What does this discussion suggest about the relationship between the determinism of a physical system and the predictability of that system?

#### **D. Introducing Chaos**

1. Stewart introduces the reader to chaotic behavior in the first chapters of his books. How does he do it?
2. He illustrates his introductions to chaos by describing the behaviors of particular dynamical systems or models of dynamical systems. What are those illustrative examples?
3. Is this a good system with which to illustrate chaotic behavior? Why?
4. In what respects are systems governed by discrete mappings and systems governed by differential equations similar? In what respects are they different?

### **STEWART, CHAPTER 2: EQUATIONS FOR EVERYTHING**

#### **A. Models of the Solar System**

1. Geocentric models.
  - a. Ptolemy.
  - b. Antikythera mechanism.
2. Heliocentric models.
  - a. Copernicus
  - b. Kepler's laws.
3. Tychonic model.
4. What does Stewart leave out of this account of models of the solar system?
5. What is the basis on which one would prefer one model to the others?

#### **B. Dynamics**

1. Galileo.
  - a. Falling bodies.
  - b. The pendulum.
  - c. Galilean satellites of Jupiter; Kepler's third law.
2. Newton.
  - a. Laws of motion.
  - b. Law of gravitation.
  - c. Calculus.
    - (i) Differentiation.
    - (ii) Integration.

#### **C. Analytical Dynamics**

1. Vibrations.
  - a. Strings.

- b. Bells.
  - c. Drums.
  - d. Organ pipes.
2. Fluid dynamics.
  3. Flow of heat.
  4. How would mathematicians have studied such systems and determined their behaviors?
  5. What is the paradigm for doing classical physics?

**D. Other Issues**

1. Often, solutions cannot be found exactly and in closed form.
2. Technical problems.
  - a. Three-body collisions.
  - b. Singularities.
3. Lagrangian and Hamiltonian formulations of mechanics.
4. Statistical problems.

**E. Questions.**

1. The chapter is essentially a history of astronomy and physics. What are the highlights?
2. What are the attributes of classical physics that are portrayed in the chapter?
3. What limits our power to study physical systems in this way?
4. What is the role of mathematics in all of this?
5. Why is the Copernican model of the solar system preferable to the Ptolemaic model?
6. What are the issues involved in the predicting the motion of a single planet around a star? (The two-body problem.)

**F. Elements of Calculus**

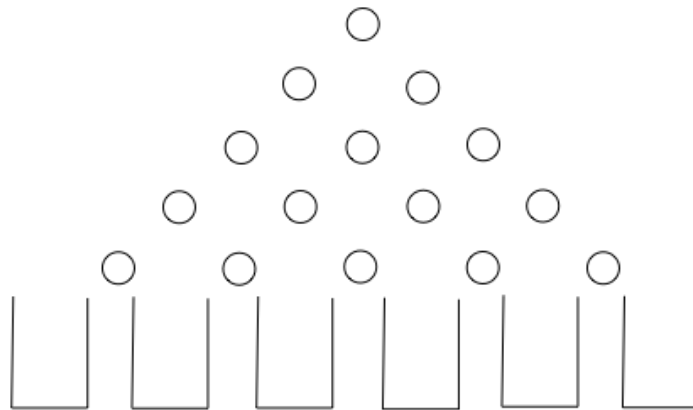
1. Differentiation.
  - a. Plot a curve representing a function.
  - b. Plot a straight line tangent to the curve at a point.
  - c. Define the slope of the line.
  - d. The derivative at the tangent point is the slope of the line.
2. Integration
  - a. Plot velocity against time.
  - b. Approximate the velocity curve in terms of line segments.
  - c. Estimate the displacement of the particle that accumulates in a given interval of time as the sum of the areas under the line segments.
  - d. Claim that we can improve the estimate by taking smaller intervals of time and a larger number of

(smaller) line segments.

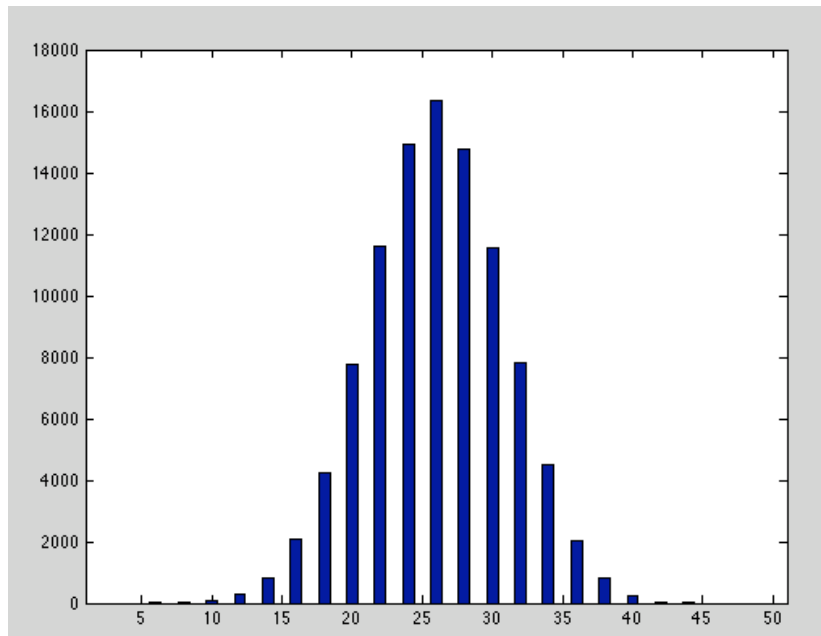
- e. In the limit, the accumulated displacement is the area under the velocity curve.
3. The point is not to do the mathematics. The point is to understand the claim that is based on doing the mathematics.

### **STEWART, CHAPTER 3: THE LAWS OF ERROR**

In science museums, e.g. The Museum of Science and Industry, one often encounters a mechanical illustration of statistics with the aid of the following apparatus. Pegs are mounted in rows on a vertical board. The top row contains one peg, the second row two pegs, the third row three pegs, and so on, so that the pegs are arranged in a triangular array. A vertical channel leads to the top peg from above, and a row of bins lies below the base of the triangle that is formed by the lowest pegs. For the benefit of the viewer, the front wall of the apparatus is a sheet of Plexiglas. A sketch of the apparatus is shown below.



The elementary experiment to be performed with this device consists of dropping a ball into the channel leading to the top peg and observing the bin in which the ball arrives after it makes its way downward through the array of pegs. In a museum demonstration, many balls are dropped successively into the upper channel, and the viewer is asked to consider the relative numbers of balls that arrive in the different bins. The results of this demonstration are shown below



**A. Analysis of the Apparatus as a Deterministic System**

1. Consider the elementary experiment in which we introduce a single ball into the apparatus. Is this a deterministic dynamical system? Explain.
2. If you have concluded that it is not a deterministic system, then explain where the randomness arises.
3. If you have concluded that it is a deterministic system, then is it chaotic in a technical sense?

**B. Analysis of the Apparatus in Terms of Probability and Statistics**

1. What aspect of the behavior of the system might we describe in probabilistic terms? What values would you assign to the relevant probabilities?
2. What aspects of the museum demonstration with this apparatus would one describe in statistical terms.

**STEWART, CHAPTER 4: THE LAST UNIVERSALIST**

**A. Equilibrium and Stability**

1. Consider a bowl of the form of a hemisphere and a ball bearing free to roll about in the bowl. Describe a situation in which the bearing is in static equilibrium (i.e., at rest). Is this a stable state of equilibrium? How can you tell?
2. Now turn the bowl upside down. Again the ball bearing is free to roll about on the inverted bowl. Is there a static equilibrium state in which the bearing is at rest? Describe it. Is this a stable state of

equilibrium? How can you tell?

3. Now turn the bowl upright. If there were no friction or air resistance, then we could get the bearing to roll at a constant rate around the bowl on a horizontal circular trajectory. This would be a state of dynamical equilibrium in which the force of gravity, the centrifugal force, and the force exerted by the bowl just balance to keep the bearing in a steady motion. How would one test this state for stability? If the system were stable, then what would happen? What would happen if it were not stable?

#### **B. Stability of the Solar System**

1. Imagine that we could investigate the stability of the solar system experimentally with the aid of a time machine. In order to perform the experiment, we visit the solar system two billion years in the future. Describe the arrangement and motions of the planets that you would expect to observe if the solar system were stable.
2. Describe the arrangement and motions of the planets that you might expect to observe if the solar system were unstable.
3. Is this concept of the stability of the solar system the same as the concept considered above of the stability of states of the ball bearing free to roll on the surface of the bowl?

#### **C. Poincare's Methods**

1. In page 59, Stewart explains that Poincare represents the state of a dynamical system in terms of a point in "some huge-dimensional phase space." Moreover, the motion of the system is represented by a curve traced out by that point in the phase space. Consider a single planet moving in the gravitational field of the sun. Describe the phase space in which Poincare would represent the motion of the planet. How many dimensions would that phase space have? What are those dimensions?
2. What are the principles or laws that determine the curve in the phase space representing the motion of the system considered in the preceding section?
3. Stewart then describes the use of a "Poincare section" in order to find periodic orbits of the system. For the case considered above of a single planet, describe a possible Poincare section. Suppose we could watch the point representing the state of the system move about in the phase space. How might we use the Poincare section in order to decide whether or not the motion is a periodic

orbit.

4. Stewart describes "Hill's reduced model" in terms of "Neptune, Pluto, and a grain of interstellar dust." What seems to be missing in this picture?
5. What are the "footprints of chaos" that Poincare found when he investigated a surface of section for Hill's reduced problem?

## **STEWART, CHAPTER 5: ONE-WAY PENDULUM**

### **A. Dynamics of a Pendulum Without (Much) Mathematics.**

1. How well can we understand Figures 27 and 29 without making use of mathematical formulae? In other words, how well could we figure out how to draw Figures 27 and 29 without making use of mathematical formulae?
2. One method: Watch a real pendulum (possibly in the mind's eye) and draw graphs and figures that capture the important qualitative features of the motion.
3. Another method: Think about the consequences of Newton's second law of motion. What is the force and resulting acceleration? How does the acceleration change the velocity? How does the velocity change the position?
4. These are qualitative methods. They do not enable us to draw accurate graphs and figures. They do enable us to understand accurate graphs and figures.
5. Consider an ideal pendulum of the kind described by Stewart. In other words, a mass at the end of a massless, rigid rod, which is attached at the other end to a frictionless pivot. The angle  $\alpha$  between the rod and a vertical line defines the "position" of the pendulum. (We may find it helpful to make a sketch of the pendulum and label the angle  $\alpha$  in various situations.) By convention, the angle  $\alpha$  is positive when the mass is displaced to our right and negative when the mass is displaced to our left.
6. If the angle  $\alpha$  represents the position of the pendulum, how is the corresponding velocity defined or described?

### **B. A Standard Description of the Motion.**

1. The goal here is to plot graphs of the position and velocity

at different times.

2. Visualize the motion of the pendulum in the case that the amplitude of the oscillation is small. Sketch a plot of the position  $a$  of the pendulum against time.
3. Sketch a plot of the velocity of the pendulum against time. Line up the plots of the position and velocity vertically. (Hints: At what positions  $a$  does the velocity vanish? Likewise, at what positions  $a$  does the velocity have its greatest magnitude?)
4. In what respects does the bottom curve in Figure 27 differ from the other curves? How might we explain those differences?
5. Now visualize the motion of a one-way pendulum. Sketch a plot of the velocity against time. Mark points on the plot where the position of the pendulum is straight down ( $\theta = 0$ ) and straight up ( $\theta = \pi$ ). Finally, sketch a plot of the position  $a$  against time.
6. Now try to visualize the motion of a pendulum in the critical case that just separates the cases of one-way motion and two-way motion. Sketch plots of the position and velocity against time.

### **C. Description of the Motion in Terms of Phase Portraits**

1. Without referring to the formula on page 70 for the energy of a pendulum, explain the qualitative appearance of the curves in Figure 29 that represent two-way motions of the pendulum. For example, try to explain the qualitative appearance of those curves as consequences of Newton's second law of motion.
2. Likewise, explain the qualitative appearance of the curves in Figure 29 that represent one-way motions of the pendulum.
3. Sketch the curves in Figure 29 that would represent the critical case of motions of the pendulum that just separate the cases of two-way motion and one-way motion.
4. Consider the versions of the phase portrait of a pendulum in Figures 32 and 33. Imagine that there is a mechanism that pumps energy into the pendulum. (For example, stand to the

right of the pendulum and blow on it every time you see it moving to the left. Sketch or describe the trajectory of the pendulum. Is there any ambiguity about the future motion of the pendulum?

#### **STEWART, CHAPTER 6: STRANGE ATTRACTORS**

1. What is the essential question about the behavior of dynamical systems that is addressed in this chapter? What is the answer for a two-dimensional system (i.e., for motion in a plane)?
2. Stewart also explains that Smale began with an incorrect conjecture. What is the essential content of that conjecture as portrayed by Stewart?
3. In what sense does the analysis of the van der Pol oscillator (Stewart, pp. 91-92, 137) constitute a mathematical proof that Smale's conjecture is wrong?

#### **STEWART, CHAPTER 7: THE WEATHER FACTORY**

1. On page 117, Stewart indicates that the prediction of weather with the aid of a computer involves a "double approximation." What are the two approximations that are meant here?
2. According to the Guardian article reproduced on pages 119-120, the Meteorological Office was to investigate the failure of Cyber 10 to predict the hurricane of October 15, 1987, in southern England and to "find out what went wrong." What, if anything, did go wrong?
3. According to the same article, Cyber 10 did predict violent storms in the North Sea on the day of the hurricane in southern England. In what sense can we interpret the contrast between the weather predicted and the weather observed as a manifestation of chaos?
4. What is thermal convection? What is the arrangement of a system in which one would demonstrate thermal convection in a laboratory? What are the conditions under which convection would occur? What are the physical processes that govern the conditions under which convection occurs? Which of the physical processes inhibit convection and which drive convection? Under what conditions is convection chaotic?

5. On pages 122-126, Stewart describes Lorenz's investigation of thermal convection. Is this description complete? What has been left out?
6. What is the butterfly effect? How does it differ from the prevailing expectations at the time of the discovery?
7. On pages 133 & 134, Stewart argues that the Lorenz attractor is similar to strange attractors described in Chapter 6. In what respects are these attractors similar?

#### **STEWART, CHAPTER 8: RECIPE FOR CHAOS**

1. What is the physical system represented by the Smale's horseshoe constructed on pages 137-139? What are the physical laws that underlie the equations governing the behavior of that system?
2. What aspects of chaotic behavior are represented or modeled by Smale's horseshoe?
3. What is the Henon-Heiles model and the Henon map?
4. What is the logistic map?

#### **General Discussions**

1. Go over the following concepts about phase space: sources, sinks, saddles, limit cycles and attractors.
2. What is this Cantor cheese.
3. Explain what is meant by phase space and trajectories in phase space.
4. What is a Poincare section and why is it useful?
5. Stewart ends his discussion of the "over the top" pendulum by alluding to the topology of a "U-tube". How can the behavior of the pendulum can be described by trajectories on the U-tube.