Deterministic Chaos

I. INTRODUCTION - ORDER VS CHAOS

One of the most mysterious aspects of the natural world is the coexistence of order and disorder. Some things appear to be fairly predictable. These things appear to obey fairly clear, rigid rules. When you flip the light switch the lights come on (mostly). When you turn the key in car's ignition, the engine starts (mostly). The sun rises at a very precisely predictable time each morning and sets at an equally predictable time each evening. When you throw a ball in the air it comes back down. As a species, we are able to build dams and roads and bridges; we are able to place communication satellites into orbit; we are able to tame (much of) our environment. All of these things are predictable, obey rules, are orderly, and can be controlled.

Then there are other things: the weather, the stock market, some sporting events, games of "chance," outcomes of some elections, and so on. These things are much less predictable, and certainly not controllable. We often say that they contain aspects of **randomness**, meaning no rhyme-or-reason, no predictability. Random is the opposite of deterministic. In randomness, future events are not *rigidly* determined by past events. In **deterministic** behavior, the future *is* rigidly determined by the past. Newton's Laws of Motion are an exquisite example of determinism. Once the initial state of a Newtonian system is known, the future is completely determined by the rules that relate state (position and velocity) to the forces acting.

Why is the universe filled with some stuff that's orderly, deterministic, Newtonian, and other stuff that's not? And how do we know when a system is one kind or the other? There is a growing awareness that some of the irregularity in nature is **not** due to randomness. In the last 30 years or so, it has become apparent that some apparently erratic and seemingly unpredictable behavior is actually deterministic. That is, the behaviour is determined by a well defined set of rules. The emergence of irregularity from order is called **deterministic chaos**. (Deterministic chaos has been brought to the attention of large numbers of nonscientists through a popular book *Chaos* by James Gleick and via Jeff Goldblum's character in the movie of Michael Chrichton's *Jurassic Park*.)

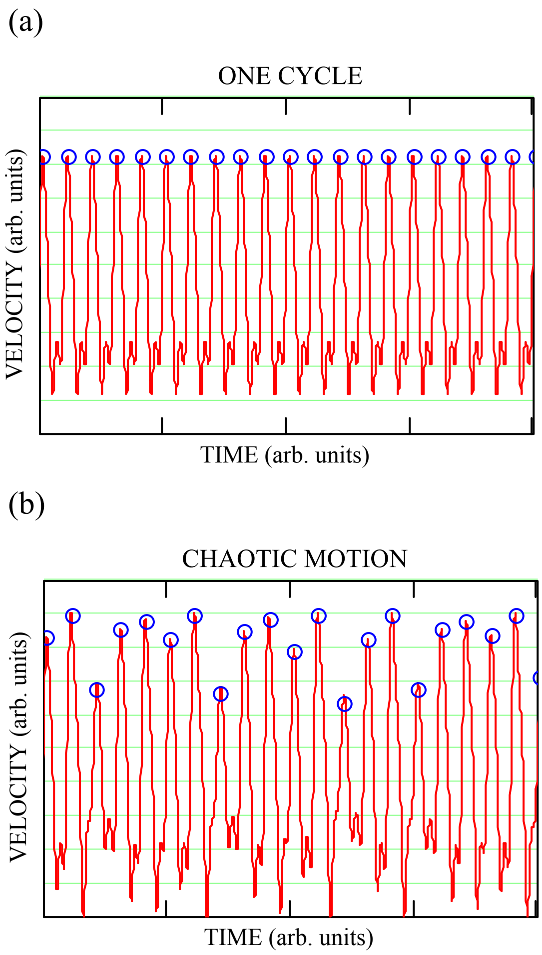

Figure 1. Motion of a damped, driven pendulum. (a) Periodic motion at the frequency of the driving force for a particular value of F_0. (b) Chaotic (nonperiodic) motion of the pendulum for a higher value of F_0.

II. NONLINEAR VS LINEAR SYSTEMS

A most basic requirement for the existence of chaos is that the dynamics of the system be **nonlinear**. To illustrate what this means, we can compare the driven damped harmonic oscillator (DDHO), which is a **linear** system with a driven damped pendulum (DDP), which is a nonlinear system.

The equation of motion for the DDHO, which we have previously studied, can be written as

$$ma_x + bv_x + kx = F_0 sin(\omega_d t). \qquad (1)$$

This equation looks slightly different than before: the ma_x term, the spring force term, and the damping force term have all been moved to the left of the equation (by subtraction from both sides of the equation), while the driving force remains on the right. The equation of motion has been written in this manner so that the dependent dynamical variable x and its time derivatives, $v_x = dx/dt$ and $a_x = d^2x/dt^2$, all appear on the left hand side. This equation is a linear equation because x and its derivatives appear only to the *first power*. That is, they don't for example appear squared (x^2) or cubed (${v_x}^3$) or in some more complicated function (e^x, e.g.). Because it is linear, this system will never exhibit chaos.

Let's now consider a DDP, which is a simple pendulum with some built-in friction and also a driving force. Instead of a spring the restoring force is due to gravity. Further, instead of x, the dependent dynamical variable is θ, the angle of the pendulum from the vertical. Similar to v_x and a_x in Eq. (1), the equation of motion for the DDP involves time derivatives of θ, $v_\theta = d\theta/dt$ and $a_\theta = d^2\theta/d\theta^2$, and can be written as

$$mla_\theta + blv_\theta + mgsin(\theta) = F_0 sin(\omega_d t). \tag{2}$$

The terms in this equation directly correspond to those in Eq. (1). This first term is the inertia term (m = mass, l = length of the pendulum), the second the friction term, the third term the restoring force (notice the mg), and the term on the right hand side is the external driving force. You may have already noticed that the position variable θ does not appear linearly, but is the argument of the $sin(\theta)$ function. Thus, this equation of motion is nonlinear.

Because of this nonlinearity, we have the *possibility* of observing chaos in the DDP. Let's first remind ourselves of what the **attractor** of the linear DDHO system is like.[1] Recall, for that oscillator the attractor is characterized by **periodic** motion at the same frequency as the driving frequency. In other words, the motion repeats itself at the frequency $\omega_d/(2\pi)$. For the pendulum, we can also get the same sort of motion if the amplitude F_0 of the driving force is not too large. This is illustrated in Fig. 1(a), which shows the velocity (v_θ) of the pendulum as a function of time for a certain (small) value of F_0.[2] The exact periodic nature of the motion is emphasized by the circles that mark the velocity v_θ at intervals separated by the driving-force period $T_d = 2\pi/\omega_d$. However, there are certain (larger) values of F_0 that produce a very different kind of motion – chaos! Figure 1(b) illustrates this. Although the motion has oscillations that have a large component at the driving frequency, closer scrutiny reveals that the motion is, in fact, **nonperiodic**: *the motion never repeats itself.* This is can most clearly be seen by noting that the sequence of amplitudes of the circles in Fig. 1(b) never repeat themselves.

To sum up what we have seen for motion of the DDP, we can say that for a small amplitude driving force the motion is periodic at the frequency of the driving force, but for at least some larger amplitudes of the driving force the motion exhibits chaos. Keep in mind though, that even there is no repetition in the motion of Fig. 1(b) the motion is as deterministic as the periodic motion shown in Fig. 1(a).

III. THE PERIOD DOUBLING ROUTE TO CHAOS

If this were all that there is to chaos, it would be interesting, no doubt. However, one of the things that makes the study of chaos interesting is how it develops as parameters of the system are changed. As we have just seen, for one value of F_0 the motion is periodic at the driving frequency, and for another value of F_0 the motion is chaotic. So one might ask, what happens to the motion if we look at a range of values of F_0? Do we simply observe periodic motion [as shown in Fig 1(a)] at some values of F_0 and chaos [Fig. 1(b)] at other values, or are other behaviors possible? Well, as it turns out, there is an often observed sequence of dynamical motions know as the **period doubling route to chaos**.

Before discussing period doubling we need to define a term generic to chaos – the **control parameter**. Control parameters are any parameters of the system that can, in principle, be adjusted to change the dynamics of the system. The pendulum has many control parameters: the mass m, pendulum length l, damping parameter b, gravitational acceleration g (remember this is only a calculation!), driving frequency ω_d, and driving-force amplitude F_0. In the section above we varied one of these parameters, the driving-force amplitude F_0, to investigate the dynamic of the system. In the rest of this discussion we will continue to vary F_0 to study the DDP.

So what is the period doubling route to chaos? We illustrate it in Fig. 2, which plots the attractor velocity vs time as F_0 is increased through a range of values. We first start with a value where we observe the simple periodic motion, previously illustrated in Fig. 1(a). If F_0 is slightly increased then at some particular value we observe a new type of periodic motion where *the period of the motion is twice as long as the driving period* – period doubling! This is shown in the graph labelled "**TWO CYCLE**" in Fig. 2.

But this is only the beginning of the period-doubling route to chaos. For if we carefully adjust F_0 to be slight larger than for the two cycle we obtain another doubling of the period – a **four cycle**! This is also illustrated in Fig. 2. Now this is beginning to get interesting! We can do the same thing again: if we carefully increase the driving parameter F_0 we obtain an eight cycle, then a 16 cycle, 32 cycle, 64 cycle, ... In fact, if we could carefully enough increase F_0, we in principle could observe any 2^n cycle (where n is any integer). However, at some point while increasing F_0 we eventually hit a value of F_0 that produces chaos, as shown in both Fig. 1(b) and the last graph in Fig. 2. In a sense chaos is simply a 2^n cycle where n is infinitely large!

Summarizing to this point, many dynamical systems exhibit chaos. In thinking about chaos there are several concepts to keep in mind. First, nonlinearity in the equations of motion is a requirement for chaos. Second, the control parameters must be in their proper ranges for chaos to be observed. The motion of the system is

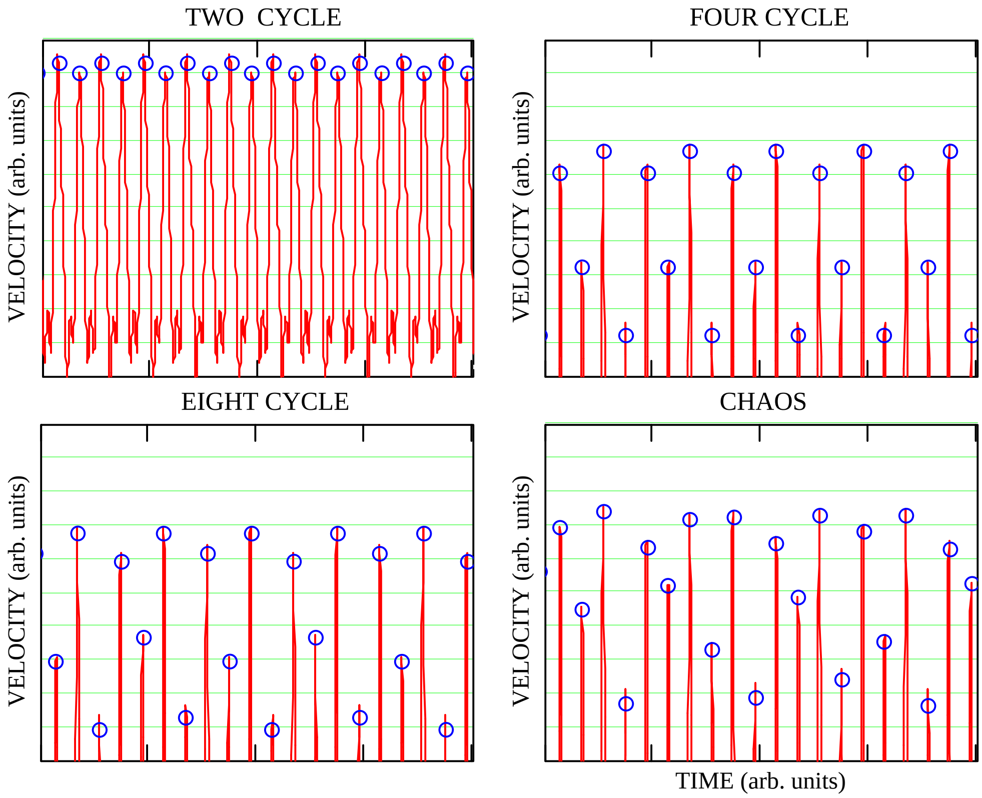

Figure 2. Velocity vs time for a driven, damped pendulum. Sequence of attractors for the oscillator as the amplitude F_0 of the driving force is increased. The two cycle graph shows the whole velocity curve. The other three graphs only show the top portion of the velocity curve.

then typically studied as a function of one of the control parameters. The period doubling route to chaos that we have seen for the DDP is rather ubiquitous: it has been observed in wide variety of apparently unrelated systems, including electrical diodes, fluid convection cells, and, as we shall now see, in models of animal population dynamics.

IV. CHAOS AND POPULATION DYNAMICS IN BIOLOGICAL SYSTEMS

The periodic doubling transition can be studied without having to get into a lot of complicated mathematics. One very simple example of a nonlinear dynamical system (**dynamical** means a system that changes in time) that shows much of the same interesting behavior comes from population biology. Many wildlife populations are known to have occurrences of regular, periodic behavior and periods of wildly fluctuating population. Figure 3 shows the variation of wolverine pelts taken in British Columbia over a period of 65 years. The fluctuations may be attributable to several factors, but the underlying population dynamics is probably one of the strongest factors.

The population dynamics model that we examine here is called the **Ricker Equation**

$$x_{k+1} = b\, x_k \exp\left(-x_k\right). \qquad (3)$$

This equation is used to model a given species' population size in a finite sized environment. In it the variable x_k is the size of the population divided by the environment's **carrying capacity** (the population size the environment can support at optimal energy consumption). The subscript k labels the generation (starting at 0 and incrementing by 1 each generation), so the Ricker equation describes how the population of each generation depends upon the previous generation. That is, x_k is the population of the k^{th} generation and x_{k+1} is the population of the next, or $k+1$ generation. In fact, the population of the $k+1$ generation only depends upon the previous (k) generation and not on, say, the $k-1$ generation. Thus the equation describes a situation where each individual lives for only one generation. (While not valid for many animals, this is an appropriate assumption for certain insects and fishes.)

In the DDP example there are several control parameters. We happened to study the system by varying the driving force amplitude F_0. In the Ricker Equation there is only one control parameter, the intrinsic birth rate b, which is the average number of births per generation per individual in the population.

When x is much smaller than 1 (that is, when the size of the population is much less than the carrying capacity), $\exp(-x)$ is approximately 1.[3] So, for small population sizes the Ricker Equation is equivalent to $x_{k+1} = b\,x_k$. As long as this approximate relation holds, it is easy to show that $x_{k+1} = b^k\,x_0$, where x_0 is some starting value. (Make sure you understand why. Start with $x_1 = b\,x_0$. Then $x_2 = b\,x_1 = b\,(b\,x_0) = b^2\,x_0$, and so forth.) If $b < 1$ (less than one offspring produced per individual), the population size steadily decreases and eventually becomes extinct. If, on the other hand, $b > 1$ (more than 1 offspring produced per individual), the population grows by a larger amount in each generation because b^k increases nonlinearly as k increases. This is called **exponential growth**. If left unfettered, exponential growth will lead to an enormous population.

That's where the "exp" term in the Ricker equation comes into play. The bigger x the smaller is $\exp(-x)$. This term describes the effects of too many individuals competing for a finite amount of energy in the environment (finite carrying capacity). You can think of it as being the probability an offspring will survive to maturity to produce more offspring. The product $b\exp(-x)$ is therefore the *effective* birth rate of each individual. The number of births per individual times the probability each birth will survive to reproduce. If the population size ever becomes too large, the species pays a price in the next generation because few offspring are produced, and the population decreases. Of course, if the population ever becomes very small, exponential growth returns it to ever-larger values. This interplay between growth and collapse is akin to the effects of the driving force and frictional force in the driven, damped pendulum.

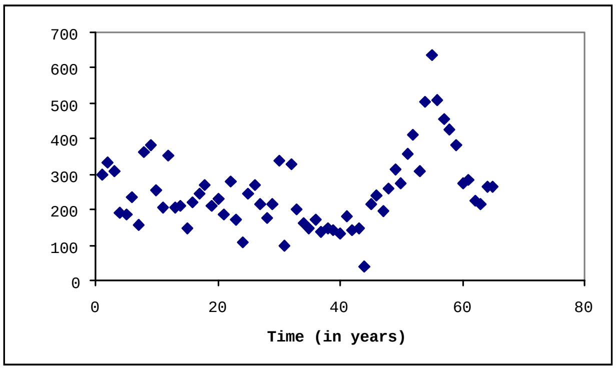

Figure 3. Number of harvested wolverine pelts per year in British Columbia over a time span of 65 years

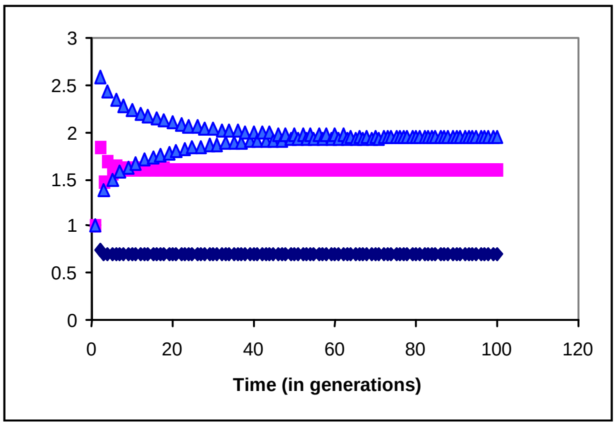

Figure 4. Population vs generation number for the Ricker Eq. with *b* = 2 (lowest curve) , *b* = 5 (middle curve), and *b* = 7 (upper curve).

V. FIXED POINTS

The interplay between the factors b and $\exp(-x_k)$ is also responsible for some very interesting results. First, note that $x = 0$ is a special value. If you put 0 in for x on the right hand side of the Ricker Equation you get 0 back out. The value 0 never changes. Such an unchanging value is called a **fixed point** of the dynamics. When $b < 1$, every starting value of x eventually evolves to 0. For $b < 1$, zero is thus known as **stable fixed point**. In the language of dynamical systems, this stable fixed point is an **attractor**. This situation is analogous to the DDP when $F_0 = 0$, which results in an attractor characterized by $v_\theta = 0$.

Zero is always a fixed point of the Ricker Equation, but when $b > 1$, it is **unstable**. That means that any slight deviation from zero will evolve away from zero, not come back.[4] What happens when $b > 1$, if zero is not stable? The answer is that the population reaches a new stable fixed point; the value of which is *not* zero. Figure 4 shows what happens for any initial value of x (except zero) and several values of the control parameter b greater than 1: $b = 2$ (bottom curve), $b = 5$ (middle curve), and $b = 7$ (top curves). Eventually, x goes to a fixed value greater than zero. The fixed value reached in each case is greater for greater values of b. Again comparing this system with the DDP, this behavior is analogous to the **one cycle** in Fig. 1(a) where the system has the same value at every sampled point (on the attractor). Note that as b increases it takes longer and longer for the dynamics to settle into the fixed-point value. This is evidence that the fixed point is getting increasingly less stable as b increases.

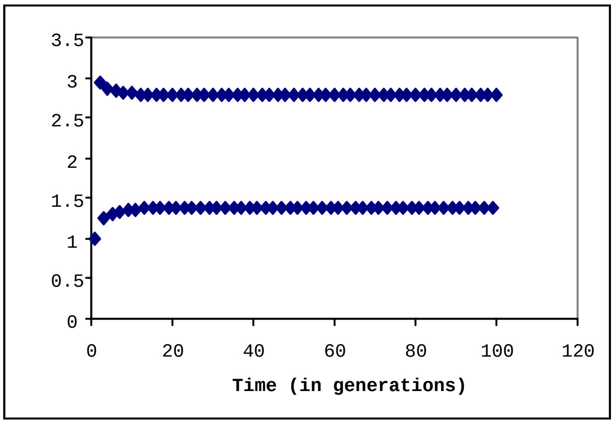

Figure 5. Illustration of a two cycle attractor in the population for *b* = 8.

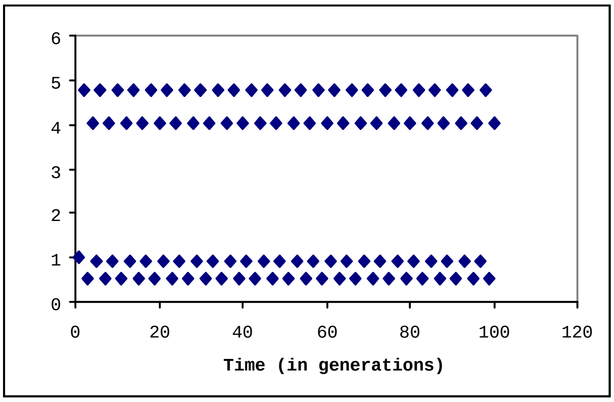

Figure 6. A four cycle attractor for the Ricker Eq. for *b* = 12.5.

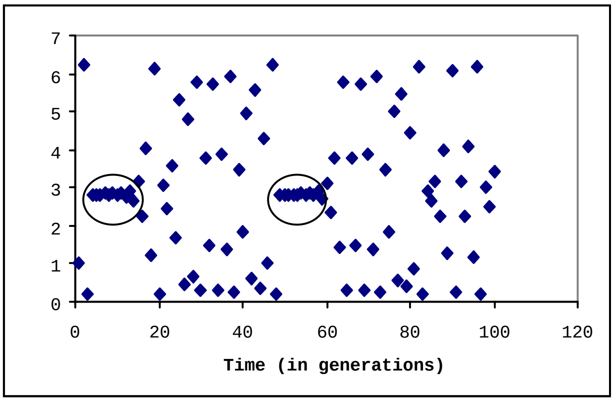

Figure 7. Chaotic dynamics for *b* = 17 in the Ricker Eq. Notice that there are two times (circled) when the dynamics come close to the unstable fixed point at ~2.83. For a short time the system stays near the fixed point before moving off to other parts of the attractor.

VI. PERIOD DOUBLING, AGAIN!

In fact, at about $b = 7.4$, the nonzero fixed point also becomes unstable. Instead, we observe a pattern of population values that repeats every other cycle instead of every cycle. That is, the repeat period has doubled, and the attractor is now a two cycle. For $b = 8$, the fixed point at zero and the nonzero fixed point still exist, but both are unstable. The stable behavior is instead a two-cycle, illustrated in Fig. 5. This behavior for the Ricker equation is analogous to the two cycle motion of the DDP shown in the first graph of Fig. 2.

When b becomes approximately 12.5, the two cycle itself becomes unstable and is replaced by a stable four cycle, as shown in Fig. 6. As with the DDP, this sequence of period doublings occurs again and again as b is made larger and larger (4 cycle goes to 8 cycle to 16 cycle to 32 cycle, and so on, infinitely many times). As before, the repeat time becomes so long (eventually infinitely long!) that the population size becomes aperiodic and the attractor is thus chaotic. An example of deterministic chaos (for $b = 17$) is shown in Fig. 7. (You may think a birth rate of 17 is pretty high but many insects and fishes produce many more progeny than that.) The big-picture point here is that Ricker Equation reproduces qualitatively all the same behavior seen in the DDP. Tat is, even though they are seemingly very different systems, there are striking similarities in the development of chaos in both systems.

VII. FIRST RETURN MAPS

You may have noticed that much of the information in Figs. 4 - 6 is rather repetitive: once the system is on the attractor the population attains only a limited number of values, depending upon the dynamics. For these n cycles and for chaotic dynamics as those shown in Fig. 7, the attractor can be presented in a more compact form with a type of graph known as a **first return map**. A first return map is simply a graph of the sampled value of the dynamical variable vs the previously sampled value of the same variable. For the DDP, this would be a plot of the circled values of the velocity (see Figs. 1 and 2) versus the just previously sampled value of the velocity. For the Ricker equation a first return map is a graph of the population of each generation vs the population of the previous generation, that is x_{k+1} vs x_k for all k such that x_k is on the attractor.

Figure 8 shows a series of first return maps for the Ricker equation. The first 3 graphs, labelled $b = 7$, $b = 8$, and $b = 12.5$, correspond to the data shown in Figs. 4 - 6. The one, two and four cycle nature of the dynamics is readily observed in the first return maps. Also shown in Fig. 8 is an 8 cycle ($b = 14.5$) and two chaotic attractors ($b = 14.9$ and $b = 17$). The last graph corresponds to the data in Fig. 7.

One important aspect of chaos that a first return map

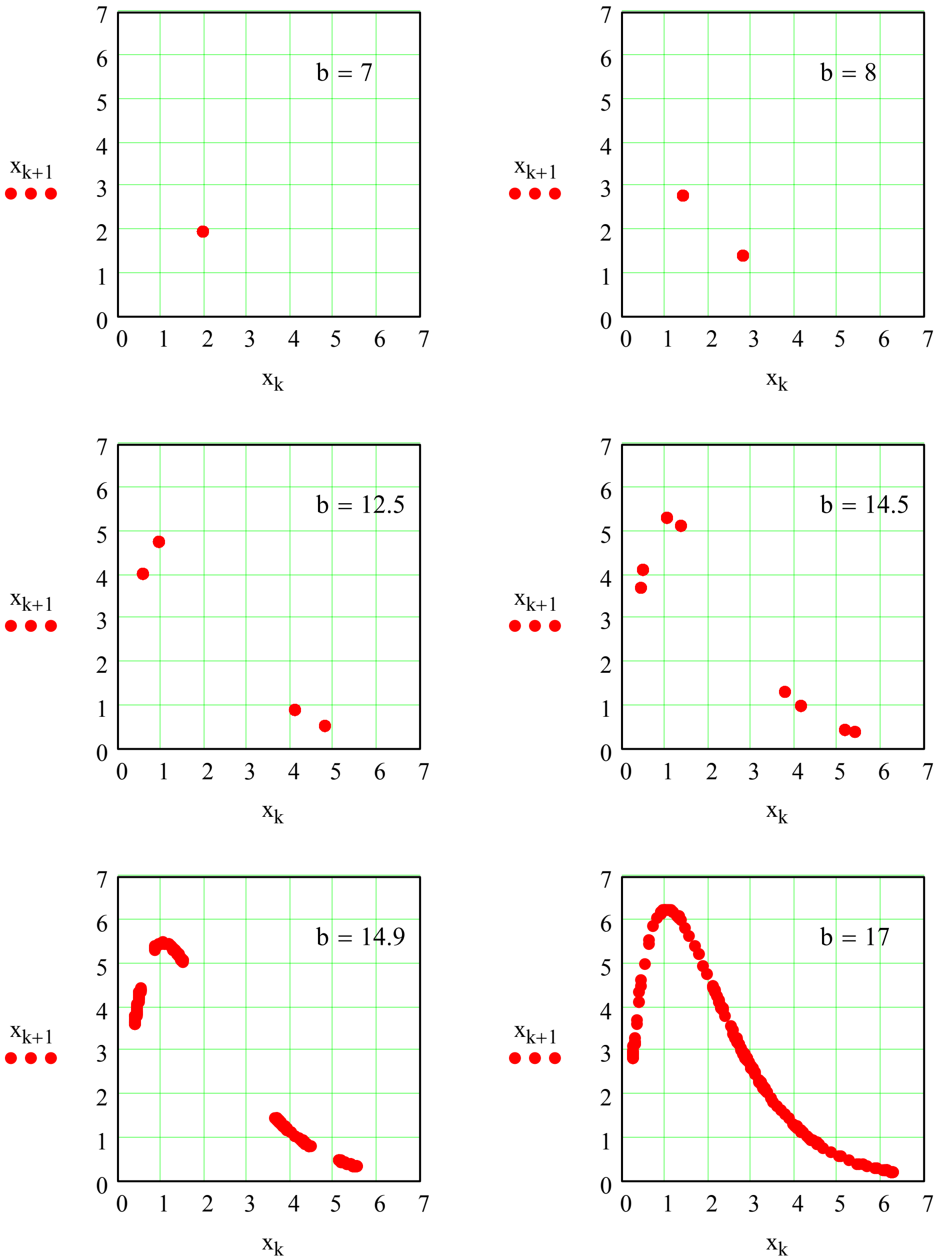

Figure 8. First return maps, which plot the population in each generation vs the population in the previous generation. 1, 2, 4, and 8 cycle attractors are shown along with two chaotic attractors.

emphasizes is the deterministic nature of chaos. For example, while there are periods of near periodicity in the data in Fig. 7, much of the data looks "random." However, when plotted as a first-return map the underlying dynamic structure (which is mathematically expressed in Eq. 3) is quite evident, as shown in the last graph of Fig. 8.

VIII. BIFURCATION DIAGRAMS

One of the easiest ways to get a global picture of the chaotic dynamics of a system is to look at another kind of graph know as a **bifurcation diagram**. A bifurcation diagram plots the attractor values (in this case the observed values of the late-time population) vs the control parameter (in this case the birth rate b). The bifurcation diagram for the Ricker equation for values of b between 6 and 20 is shown in Fig. 9. The beginning of the two

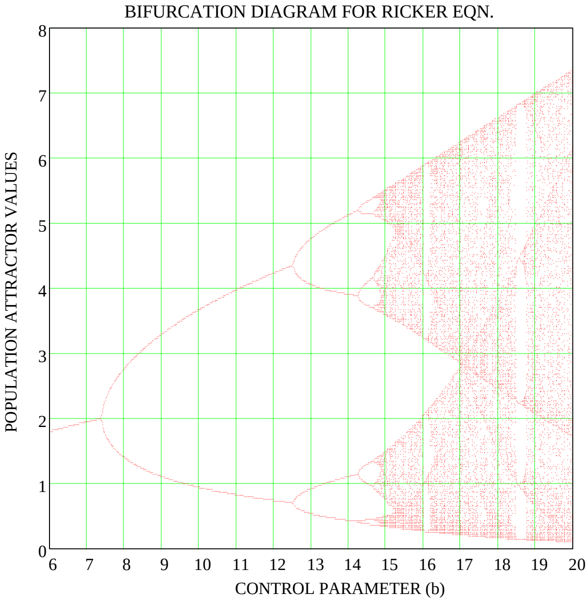

Figure 9. Bifurcation diagram for Ricker Eq. Attractor values are plotted vs the control (or driving) parameter *b*. The eight cycle is easily seen before chaos happens. Notice also there are windows at higher values of the driving parameter where *n* cycles occur: just above *b* = 16 there is a 6 cycle, and at *b* near 18.5 there is a 5 cycle.

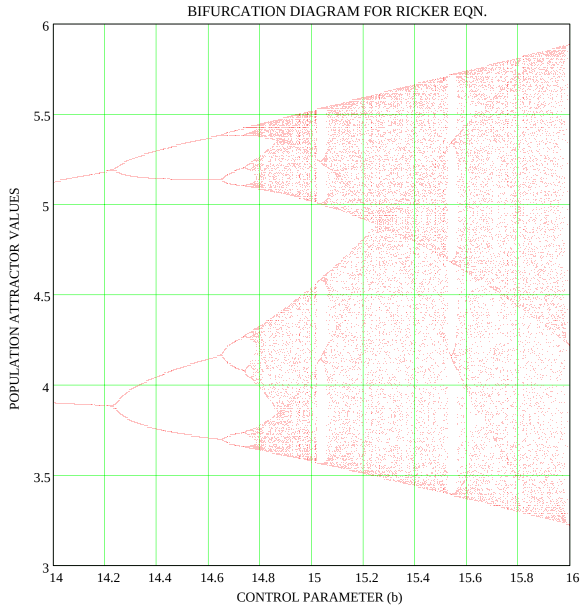

Figure 10. Zoomed in view of the bifurcation diagram for Ricker Eq. Comparing this graph with that in Fig. 9 shows the self similarity of the bifurcation diagram.

cycle is now clearly seen at b near 7.4, the beginning of the four cycle near $b = 12.5$, and so forth. The diagram also clearly shows that for any given n cycle (here n is some integer) the range of b dramatically decreases as n increases. For example, the eight cycle only occurs for b between approximately 14.2 and 14.6. On the graph the range for the 16 cycle is so small that it is not visible.

The bifurcation diagram, in addition to showing the period doubling route to chaos, also shows that there are regions at higher values of the control parameter b where chaos stops and an n cycle emerges. For example, we see that for b slightly larger than 16 there is a six cycle and for b near 18.5 there is a five cycle attractor.

One of the most fascinating things about bifurcation diagrams is their **self similarity**. This means that if we zoom in on part of the diagram (by an appropriate amount), it looks essentially identical to the original larger-scale view of the diagram. For example, in Fig. 10 we have zoomed in on the part of the diagram for b between 14 and 16 and the population (x_k) between 3 and 6. What appears to be a four cycle in Fig. 10 is actually part of the eight cycle in Fig. 9. Note that part of the sixteen cycle is also now visible in Fig. 10. In looking at this expanded portion of the diagram we see that it is self similar to the larger scale picture of the bifurcation diagram in Fig. 9: a replica of the six cycle near 16 is repeated on a smaller scale near $b = 15$, while the five cycle near 18.5 is repeated on a smaller scale at b just above 15.5. In fact this self similarity repeats at *all length scales*. That is, we could zoom in to ever smaller scales and the bifurcation diagram would continue to look the same! Such ever repeating self-similarity is one of the hall marks of a **fractal**, which is a spatial structure with a *noninteger* dimension. This bifurcation diagram is indeed a fractal!

IX. CONTROL OF CHAOS

Note the two circled stretches in Fig. 7 where the value of x seems to settle into a repeat-every-time, fixed-point pattern. The values that x takes on in each stretch are about the same. In fact, those values are almost the value of the unstable, nonzero fixed point for $b = 17$ (a value that can be calculated exactly – it's $\ln(17) = 2.833...$) This behavior is no accident (since it is completely deterministic). Although the motion is seemingly random, there are underlying (unstable) periodic behaviors that are routinely sampled. In fact, almost immediately after the second attempt to land on the unstable fixed point, the behavior attempts to be an unstable four-cycle. Do you see it?

The fact that within deterministic chaos there is a continual, recurrent, ever-failing pursuit of periodicity allows chaos to be controlled by a very subtle technique. Usual engineering control is accomplished in a "brute force" manner. The usual method for control starts with a mathematical model of the system to be controlled and

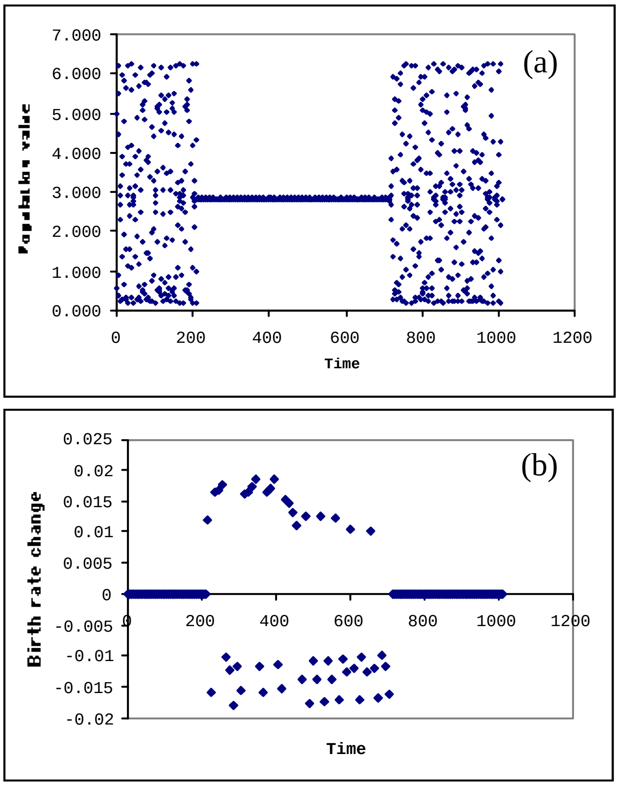

Figure 11. Control of Chaos. (a) Population vs generation number. Chaos controller is turned on generation 212 and turned off at generation 712. (b) Change in birth rate *b* needed to keep the dynamics near the stable fixed point.

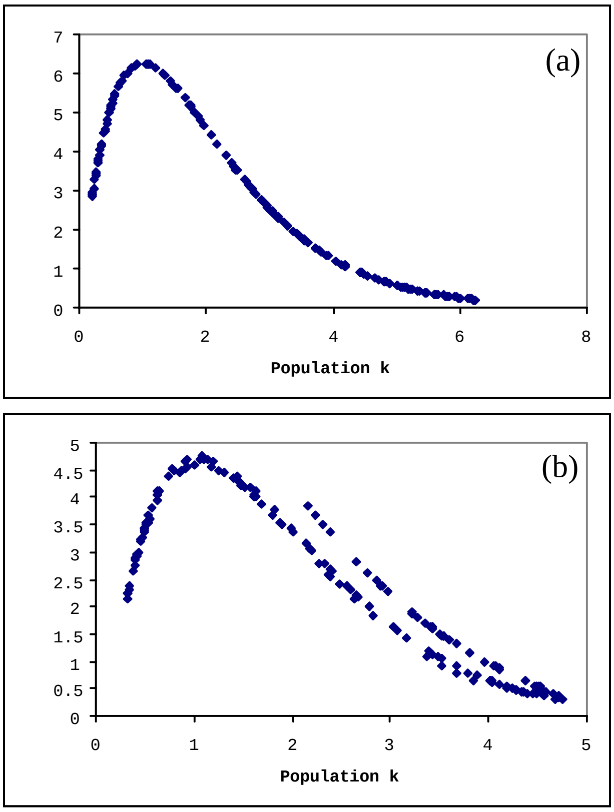

Figure 12. (a) First return map for the Ricker Eq. with *b* = 17. (b) First return map for modified Ricker Eq. with *b* = 15.9 and *c* = 0.31.

a target behavior. A controller is then designed based on the model. Often the control is energy intensive and not at all optimal. Chaos control is different. It exploits the failed periodicity character of chaos (near an unstable fixed point) and is extremely efficient. It doesn't require a model of the behavior. The way it works is as follows. You (or your computer) look for examples of recurrent unstable periodic behaviors in the output of the system (such as the two circled stretches in the previous graph). Once those are identified, you wait until the same behavior recurs (it will, though you may have to wait a bit). When that happens you change the system's control parameter ever so slightly, and only when needed, to keep the system close to the identified behavior. The details of how the parameter adjustment is actually done are fairly easy to work out, but for our purposes it is sufficient to know that little parameter jiggles are often sufficient to trap the system's behavior close an unstable fixed point.

Figure 11(a) shows control of the output of the Ricker Equation (with $b = 17$) using this method. Though *we* know the source is the Ricker Equation, the computer doesn't. Control comes on at step # 212, after an unstable fixed point value has been identified. The control stays on for 500 steps, after which it is turned off. Before the control is implemented the output is wildly

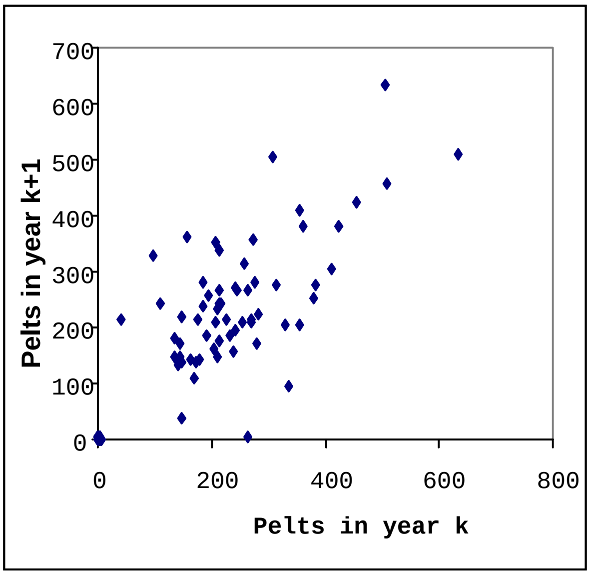

Figure 13. First return map for the pelt data shown in Fig. 1. The fuzziness of the data indicates that the dynamics are high order dynamics.

chaotic. After the control is turned off the output is again wildly chaotic. Control is maintained to within ±5% of the fixed point value by slightly adjusting the birth rate b when necessary. Figure 11(b) shows the changes in b needed to accomplish this control. Note that the changes are few (49 changes in 500 steps), and range between ±0.02 (about ±0.1% of the unchanged birth rate of 17). Very small parameter adjustments applied at just the right time can keep the system from returning to its wildly chaotic behavior. In the usual kind of brute force control, the controller would typically always be doing something and the adjustments would typically always be much larger than shown in the graph. The downside to chaos control is that you can't control the system to just any behavior. You can only control to the unstable periodic behaviors that the system permits. You can't, for example, control to a fixed point value of 1 or 5 from the Ricker Equation with $b = 17$, because the only fixed point of the Ricker Equation with $b = 17$ is about 2.83. Nonetheless, if wild chaotic swings are very undesirable (as, for example, in a wildly fibrillating heart) any control at all may be of significant practical utility.

X. SIMPLE VS COMPLEX CHAOS

To close this discussion on deterministic chaos, it is necessary to point out that chaos comes in two forms: simple (or **low-dimensional)** and complex (or **high-dimensional)**. The meaning of low- and high-dimensional can be seen in the first return map. The first return map for the Ricker equation for $b = 17$ is repeated in Fig. 12 (a). The fact that we obtain a smooth, simple curve tells us that the population in generation $k+1$ depends only on the population in generation k. When the future depends on only one value in the past the dynamics is said to be **one-dimensional**. However, if the Ricker equation is slightly modified to depend upon two previous generations

$$x_{k+1} = b\,x_k \exp\left(-x_k\right) - c\,x_{k-1}, \tag{4}$$

then the first return map (for $b = 15.9$, $c = 0.31$) looks like curve in Fig. 12(b). You can still see hints of the curve in Fig. 12(a), but the new plot shows considerable smudging out. Because in the modified Ricker Equation the population in generation $k + 1$ depends on the populations in generation k and generation $k - 1$, it is an example of **two-dimensional dynamics**.

The rule is, the higher the dimension of the dynamics the more smudged out a first return map will be. We almost always see smudged out first return maps for data collected from the real world. A typical example is the pelt data displayed earlier. Its first return map is shown in Fig. 13. Although there isn't much data, pelt yields appear to be an example of high-dimensional dynamics. We infer that almost all real systems are relatively high-dimensional.

The interesting and encouraging thing about chaos is that when it episodically tries to be periodic, the dimension of the dynamics effectively becomes lower. Thus, even if the chaotic fluctuations of a real system are high-dimensional, every once in awhile, the complexity of the dynamics spontaneously reduces. When that happens, control of even a high-dimensional system can be put into effect. There are two physiological examples of this, one involving fibrillating rabbit hearts, the other electrical activity in rat brains. In both cases the intervals between events (beating of the heart, electrical spikes in the brain) make a very blobby first return map-suggesting high-dimensional chaos. Nonetheless, in both cases episodic simplification is observed and control, using the same principles as in the Ricker example, has been experimentally implemented. There is a considerable interest in seeing whether these primitive experiments can be extended to humans with positive clinical implications such as the stabilization of fibrillation or the prevention of epileptic seizures.

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[1] Keep in mind that the attractor is the *late-time* motion of the system, which occurs when all of the initial-condition determined transients in the motion have died out.

[2] We could plot the position (θ), velocity (v_θ), or even the acceleration (a_θ) vs time to investigate the periodicity of the motion. For technical reasons we choose to plot the velocity.

[3] The term $\exp(-x_k)$ is the exponential function. That is, it is the **exponential constant** $e = 2.718281828...$ raised to the $-x_k$ power.

[4] Stability and instability can be thought of in terms of a marble and a bowl. If the bowl is open side up, the bottom of the bowl is a fixed point for a marble, because a marble placed there will stay there. It is also a stable fixed point. If you displace the marble away from the bottom a bit and let go, the marble will slide back down toward the bottom of the bowl. On the other hand, suppose the bowl is inverted, so the open side is down. Assuming the bottom of the bowl is perfectly round you can still, with *exceedingly great* care, perch a marble at the very top. But now any slight nudge will send the marble careening away from the top. In this case, the top of the bowl is an *unstable* fixed point.